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HEDGING DOUBLE BARRIERS WITH SINGLES

Abstract

Double barrier options can be statically hedged by a portfolio of single barrier knockin options. The main part of the hedge automatically turns into the desired contract along the double barrier corridor extrema. Tests of hedging performance show that (i) much of the action occurs along the lower barrier; (ii) along that barrier, fully non-automatic rebalancing may be preferred; (iii) the static hedge gives extra comfort with respect to the dynamic hedge as, after either barrier is hit, rebalancing at high volatility levels generates smooth and zero net value for comfortably large price ranges.

JEL Classification: G12, G13, C61.

Keywords: Double barrier options, single barrier options, static hedging.

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Barrier derivatives are the most liquid among the over-the-counter derivatives. Over-the-counter markets have become stronger and stronger in the industry.[1] European, continuously-monitored barrier options are European options with an American feature. Option's existence depends on whether the underlying price breaches, before or at maturity, some prespecified levels, called barriers. Given one barrier, *single knockin options* come to life and *single knockouts* expire if the barrier is hit. Given two barriers, the double barrier corridor encompasses the initial underlying price. *Double knockins* come to life and *double knockouts* expire if either barrier is hit. A portfolio of a knockin and a knockout written on the same barriers and strike is equivalent to a vanilla option with the same strike. Thus, one can focus on knockins only.

Barrier options are very popular because they are cheaper than their vanilla counterparts. This endears them to hedge funds, which thrive on achieving the biggest bang for their buck.[2] Via double barriers, investors enjoy even greater leverage potential: Single knockouts typically have barriers too close for comfort and single knockins have less knockin chances without much discount. A double knockin may be bought by a fund manager who bets against market consensus' direction but hedges her bet for marking-to-market purposes. It may also be bought by a trader who foresees a bigger volatility than the market consensus' one in both bullish and bearish scenarios.

The double barrier clause states: *if either barrier is hit*. This creates a *double barrier interdependence* and makes pricing and hedging difficult: *A double knockin is not simply the sum of two single knockins written on the corridor extrema.* I call that sum the *Basic Portfolio*. The Basic Portfolio is a super-replicating hedge: if the upper (lower) barrier is hit first, the single barrier contract written on the lower (upper) barrier contributes positive unwanted value. The hedger needs to add extra layers to get exact replication.

This work shows that, under the Black-Scholes assumptions, *double barrier interdependence commands extra hedging layers all made of single knockins with the same maturity as the double barrier knockin.*

The following numerical example shows the structure of those hedging layers. The current underlying price is \$90. Consider a double knockin call with lower barrier \$80 and upper barrier \$100. Its strike is \$90. The double knockin call[3] is priced \$12.8079. The double knockin price is mainly made of the \$100-in price (\$12.7587) as the logprice drift is positive and the probability of reaching the \$100 level first is high. The following table shows barriers, strikes, portfolio amounts,

[1]Dupont (2001) documents that, as of December 1999, over-the-counter transactions accounted for around 86 percent (\$88 trillion) of the total notional value of derivatives contracts, exchanges for about 14 percent (\$14 trillion).

[2]This generates problems because hedge fund managers have a strong incentive to drive the underlying market towards their long knockin barriers. From *The Economist*, London, March 18, 1995; Anonymous.

> ... A fierce battle between a buyer and seller of knock-in options. In late 1994 ..., Merrill Lynch, ..., and a fund managed on behalf of Micheal Steinhardt, a well-known hedge fund manager, slugged it out in the market for Venezuelan Brady bonds (repackaged debt partially backed by American Treasury bonds). The fund owned a knock-in option and was trying to push up prices by buying huge quantities of bonds. Merrill, which had sold the option, used all of its muscle to keep them below the point at which the option would have been triggered. This may explain why trading volumes in this otherwise obscure market soared: ..., at the height of the battle, some \$1.5 billion-worth of the almost \$7 billion outstanding Venezuelan Brady bonds changed hands, pushing up prices by 10%.

[3]Other option parameters are: annualized riskfree rate equal to 5%, logprice annualized volatility equal to 30%, 1-year maturity, and payout rate equal to 0.

portfolio amounts in \$s of the single knockin positions that constitute the hedging layers.

An example of Double Barrier Exact Hedge (DBEH)

Knockin barrier in \$s	Strike in \$s	Amount	Amount in \$s
381.47	343.32	0.2434	0.000015
305.18	343.32	-0.2434	-0.000023
244.14	219.73	0.3898	0.009642
195.31	219.73	-0.3898	-0.011746
156.25	140.63	0.6243	0.835973
125.00	140.63	-0.6243	-0.887747
(upper barrier) 100.00	90.00	1.0000	12.758694
(initial spot price) 90.00	(original strike) 90		
(lower barrier) 80.00	90.00	1.0000	3.757592
64.00	57.60	-1.6017	-3.667750
51.20	57.60	1.6017	0.113625
40.96	36.86	-2.5655	-0.100540
32.77	36.86	2.5655	0.000559
26.21	23.59	-4.1093	-0.000413
20.97	23.59	4.1093	0.000000
			Sum of the amounts in \$s = Double knockin price
			12.807870

The table illustrates how these single barrier options have barriers which take progressive distance from the original barrier corridor \$80 / \$100. Summing the portfolio amounts in \$s of all the first 14 single knockins (from barrier level \$21 to barrier level \$381) gives the double knockin exact price. The Basic Portfolio, the sum of a \$80-in and a \$100-in only, is priced \$16.5163. The full replicating portfolio is made of a countable infinity of single knockin positions and I call it *Double Barrier Exact Hedge* (*DBEH*). The first few positions of the DBEH are sufficient to achieve good replication of the double knockin. Rebates associated to barrier options are special cases of them so that the pricing and hedging analysis here developed embraces them.

1. Contributions

The DBEH contributes along these lines. (1) It is static. (2) It exhibits an automatically-in feature along the barriers, because it has barrier-like nature as its target contract, the double knockin. (3) It takes account of the drift towards either barrier generated by a non-trivial cost of carrying the underlying asset. (4) It establishes an explicit link between single barrier pricing and double barrier pricing. Tests of hedging performance, carried out in Section 3., suggest that (1) and (2) are the most relevant for practical purposes.

Portfolio amounts of the DBEH are static, that is, not time-varying except for the first passage time of the underlying price through either barrier. Static hedging has the advantage of suffering less from transaction costs and pricing model misspecification as you trade at most 2 times. The path-dependent options here examined often have high gammas and vegas, that is, their delta (value sensitivity to underlying price changes) is highly time-varying and option prices are quite sensitive to volatility changes. In this case, static hedging is much likely to be easier and cheaper than dynamic hedging. The first analysis of static hedging of path-dependent options is due to Bowie and Carr (1994) and Derman, Ergener, and Kani (1994). Dupont (2001) discusses the latest developments in static hedging of barrier options and applies a technique, mean-square hedging, designed to minimize the size of the hedging error when perfect replication is not possible.

Static hedging of double barrier options by means of non-barrier options has been proposed by Carr, Ellis and Gupta (1998) (CEG) and by Andersen, Andreasen, and Eliezer (2000). Along the barriers, the hedger should fully unwind the hedge because the double barrier contract automatically either comes to life or terminates. Trading along the barriers may be difficult. The main element of DBEH is the Basic Portfolio. Thus, if \$100 is reached before \$80 in due time, the \$100-in leg automatically kicks in. With the DBEH, the hedger must only unwind its non-triggered legs.

If the underlying asset commands a positive (negative) cost of carry, then its risk-adjusted price exhibits a drift towards the upper barrier (lower barrier). Even in presence of such non-trivial risk-adjusted drift, the DBEH remains exact. I show that, with zero cost of carry, the DBEH specializes to the hedge proposed by CEG. This is because, if you break down the DBEH legs into subportfolios of non-barrier options, the two hedges correspond layer by layer. The DBEH needs a countable infinity of single knockins while the hedge proposed by Andersen, Andreasen, and Eliezer (2000) handles general price-dependent volatility but needs an along-all-strikes continuum of European options and an along-all-maturities continuum of calendar spreads. I show that the cost-of-carry effect is not massive even for low levels of logprice volatility.

Single barrier option prices are well known (see Merton (1973), Cox and Rubinstein (1985), Benson and Daniel (1991), Hudson (1991), Reiner and Rubinstein (1991), Heynen and Kat (1994), Rich (1994), and Trippi (1994)). However, double barrier pricing is difficult because of the double barrier interdependence. The mathematics which unravels that interdependence is awkward, so that existing closed-form prices (Douady (1999), Hui (1996), Hui, Lo, and Yuen (2000), Kunitomo and Ikeda (1992), Lin (1997), Pelsser (2000)) achieve elegance at the expenses of financial intuition. The DBEH states that the double barrier option price is a weighted sum of single barrier option prices with weights which do not depend on the initial underlying price.

Geman and Yor (1996) and Jamshidian (1997) start from techniques based on time-horizon Laplace transforms and suggest numerical techniques for double option pricing. The analysis here develops the financial-engineering potential in those techniques by carving out explicit pricing and static-hedging results.

The rest of this work is organized as follows. Section 2 shows how the DBEH works. Sections 3 discusses its hedging performance. Section 4 concludes. The appendix gives technical details and proofs of the propositions.

2. The Double Barrier Exact Hedge (DBEH)

Here I show that, under the Black-Scholes assumptions, the double barrier option price is a weighted sum of single barrier option prices. Such pricing results cast light on the financial nature of the contract. The key feature is that they project the risk of double barrier instruments on to single barrier instruments.

Let $C^L_{knockin}(S_0, K, T)$ ($C^U_{knockin}(S_0, K, T)$) denote the price of a single knockin call with barrier L (U). The three arguments of the price function are the initial price S_0 of the underlying asset , the strike price K, and the option maturity T. The lower barrier L and upper barrier U straddle the initial underlying price S_0 and the strike K ($L \leq S_0 \leq U$ and $L \leq K \leq U$). The double knockin call, with price

$$C^{L,U}_{knockin}(S_0, K, T),$$

is a call option which is initiated whenever either the upper barrier U or the lower barrier L is touched before or at option maturity. The instantaneous return rate of the riskfree asset is the constant r and the underlying asset offers a constant instantaneous payout rate d. $C(S_0, K, T)$ denotes the standard call price.

The DBEH unravels the pricing and hedging difficulty of double barrier options in a way which makes it easily comparable with the existing double barrier option literature, in particular with the double barrier option decomposition of Carr, Ellis and Gupta (1998).

Proposition 1 *Under the Black-Scholes assumptions, the double knockin call price has the following exact decomposition:*

$$C^{L,U}_{knockin}(S_0, K, T) = \qquad \text{('Double Barrier Exact Hedge (DBEH)')}$$

$$C^{U}_{knockin}(S_0, K, T) + C^{L}_{knockin}(S_0, K, T) + \qquad \text{('Basic Portfolio')}$$

$$\sum_{n=1}^{\infty}\left(\frac{m_{BS}(0, \ln L, \ln U)}{m_{BS}(0, \ln U, \ln L)}\right)^{+n} \times \left(\frac{U}{L}\right)^{+2n} \times C^{L\left(\frac{U}{L}\right)^{-2n}}_{knockin}\left(S_0, K\left(\frac{U}{L}\right)^{-2n}, T\right) -$$

('U-First-L-In Portfolio (UFLI), Part I')

$$\sum_{n=1}^{\infty}\left(\frac{m_{BS}(0, \ln U, \ln L)}{m_{BS}(0, \ln L, \ln U)}\right)^{+n} \times \left(\frac{U}{L}\right)^{-2n} \times C^{L\left(\frac{U}{L}\right)^{+2n}}_{knockin}\left(S_0, K\left(\frac{U}{L}\right)^{+2n}, T\right) +$$

('U-First-L-In Portfolio (UFLI), Part II')

$$\sum_{n=1}^{\infty}\left(\frac{m_{BS}(0, \ln U, \ln L)}{m_{BS}(0, \ln L, \ln U)}\right)^{+n} \times \left(\frac{U}{L}\right)^{-2n} \times C^{U\left(\frac{U}{L}\right)^{+2n}}_{knockin}\left(S_0, K\left(\frac{U}{L}\right)^{+2n}, T\right) -$$

('L-First-U-In Portfolio (LFUI), Part I')

$$\sum_{n=1}^{\infty}\left(\frac{m_{BS}\left(0,\ln L,\ln U\right)}{m_{BS}\left(0,\ln U,\ln L\right)}\right)^{+n}\times\left(\frac{U}{L}\right)^{+2n}\times C_{knockin}^{U\left(\frac{U}{L}\right)^{-2n}}\left(S_0,K\left(\frac{U}{L}\right)^{-2n},T\right),$$

('L-First-U-In Portfolio (LFUI), Part II')

where the constant σ is the local volatility of the underlying logprice. The portfolio-weight factors are

$$m_{BS}\left(0,\ln L,\ln U\right)=e^{+(\ln L-\ln U)\left(-\frac{r-d-\frac{1}{2}\sigma^2}{\sigma^2}\right)}e^{+|\ln L-\ln U|\left(-\frac{\left|r-d-\frac{1}{2}\sigma^2\right|}{\sigma^2}\right)}$$

and

$$m_{BS}\left(0,\ln U,\ln L\right)=e^{+(\ln U-\ln L)\left(-\frac{r-d-\frac{1}{2}\sigma^2}{\sigma^2}\right)}e^{+|\ln U-\ln L|\left(-\frac{\left|r-d-\frac{1}{2}\sigma^2\right|}{\sigma^2}\right)}.$$

$m_{BS}\left(\lambda,x_0,b\right)$ is the moment generating function of the risk-adjusted logprice's first exit time through some barrier b once it starts from the initial level x_0. λ ($\lambda\geq 0$) is the moment generating function parameter.

Proof. See the appendix. ■

Notice that Part II of LFUI dominates in absolute value part I of UFLI. They have the same portfolio amounts, same strikes, but LFUI has higher down-in barriers than UFLI. On the other hand, Part I of LFUI is dominated in absolute value by Part II of UFLI. They have the same portfolio amounts, same strikes, but UFLI has lower up-in barriers than LFUI. Given that the original strike is within the double barrier corridor, Part II of UFLI actually consists of vanilla call options because its up-in barriers are lower than their corresponding strikes. Table **I** displays the structure of the DBEH.

Portfolio amounts and single barriers are fully characterized in terms of the risk-adjusted probability of the price ever travelling the distance $[L,U]$ from L to U and in the opposite direction, $m_{BS}\left(0,\ln L,\ln U\right)$ and $m_{BS}\left(0,\ln U,\ln L\right)$. Indeed, these two excursion probabilities make the portfolio weights. The factor $\left(\frac{U}{L}\right)^{-1}$ rescales the single knockin option prices, their strikes, and their barriers. $\left(\frac{U}{L}\right)^{-1}$ would be the risk-adjusted probability of the price ever travelling from L to U and in the opposite direction if the risk-adjusted price had zero local drift. Zero local drift for the underlying asset implies zero cost of carry and this is a natural assumption only for forwards).

Proposition 2 *Under the Black-Scholes assumptions and with zero cost of carry, the static hedge proposed by CEG and the DBEH coincide in every respect.*

Proof. See the appendix. ■

Table **II** illustrates the equivalence between the two hedges in the case of zero carrying costs. Since they corrispond layer by layer, their hedging architecture is the same. CEG conveniently represents each UFLI knockin position with one non-barrier (less exotic) option position but represents each LFUI knockin position with three non-barrier option positions.

A. Hedging Architecture

If the upper (lower) barrier is hit first, the single barrier contract written on the lower (upper) barrier contributes positive unwanted value. Figure **1** quantifies such unwanted value. Much of the

Table **I**: **The Double Barrier Exact Hedge (DBEH)**

E^Q denotes expectation under the risk-adjusted probability measure and T_U (T_L) is the first time the underlying price reaches the upper barrier U (L). The arguments of the option price functions are the underlying asset price (S_0 is the current underlying price), the strike price K, and the time to maturity, T. $C^{L,U}_{knockin}$ denotes the price of a double knockin call with upper barrier U and lower barrier L. r is the risk-free rate and d is the asset's payout ratio. σ is the local volatility of the underlying logprice. r, d, and σ are constant.

$$C^{L,U}_{knockin}(S_0, K, T) =$$

$$+C^{U}_{knockin}(S_0, K, T) + C^{L}_{knockin}(S_0, K, T)$$

Basic Portfolio

$$-E^Q\left(e^{-rT_U} 1_{\{T_U<T_L\}} C^{L}_{knockin}(U, K, T-T_U) \mid S_0\right)$$

U-First-L-In (UFLI)

$$-E^Q\left(e^{-rT_L} 1_{\{T_L<T_U\}} C^{U}_{knockin}(L, K, T-T_L) \mid S_0\right).$$

L-First-U-In (LFUI)

$$-E^Q\left(e^{-rT_U} 1_{\{T_U<T_L\}} C^{L}_{knockin}(U, K, T-T_U) \mid S_0\right) =$$

$$+\sum_{n=1}^{\infty} \underbrace{e^{+2\left(\frac{\left(r-d-\frac{1}{2}\sigma^2\right)}{\sigma^2}+1\right)n(\ln U - \ln L)} \times C^{L\left(\frac{U}{L}\right)^{-2n}}_{knockin}\left(S_0, K\left(\frac{U}{L}\right)^{-2n}, T\right)}_{\text{(single barrier } (L(\frac{U}{L})^{-2n}) \text{ down-and-in calls with barrier below the strike } (K(\frac{U}{L})^{-2n}))}$$

$$-\quad\sum_{n=1}^{\infty} \underbrace{e^{-2\left(\frac{\left(r-d-\frac{1}{2}\sigma^2\right)}{\sigma^2}+1\right)n(\ln U - \ln L)} \times C^{L\left(\frac{U}{L}\right)^{+2n}}_{knockin}\left(S_0, K\left(\frac{U}{L}\right)^{+2n}, T\right)}_{\text{(single barrier } (L(\frac{U}{L})^{+2n}) \text{ up-and-in calls with barrier below the strike } (K(\frac{U}{L})^{+2n}) = \text{ standard calls)}},$$

$$-E^Q\left(e^{-rT_L} 1_{\{T_L<T_U\}} C^{U}_{knockin}(L, K, T-T_L) \mid S_0\right) =$$

$$+\sum_{n=1}^{\infty} \underbrace{e^{-2\left(\frac{\left(r-d-\frac{1}{2}\sigma^2\right)}{\sigma^2}+1\right)n(\ln U - \ln L)} \times C^{U\left(\frac{U}{L}\right)^{+2n}}_{knockin}\left(S_0, K\left(\frac{U}{L}\right)^{+2n}, T\right)}_{\text{(single barrier } (U(\frac{U}{L})^{+2n}) \text{ up-and-in calls with barrier above the strike } (K(\frac{U}{L})^{+2n}))}$$

$$-\sum_{n=1}^{\infty} \underbrace{e^{+2\left(\frac{\left(r-d-\frac{1}{2}\sigma^2\right)}{\sigma^2}+1\right)n(\ln U - \ln L)} \times C^{U\left(\frac{U}{L}\right)^{-2n}}_{knockin}\left(S_0, K\left(\frac{U}{L}\right)^{-2n}, T\right)}_{\text{(single barrier } (U(\frac{U}{L})^{-2n}) \text{ down-and-in calls with barrier above the strike } (K(\frac{U}{L})^{-2n}))}.$$

Table **II**: **The static hedge of Carr, Ellis, and Gupta (1998)**

The arguments of the option price functions are the current underlying asset spot price, S_0, the strike price K, and the time to maturity, T. $C^{L,U}_{knockin}$ denotes the price of a double knockin call with upper barrier U and lower barrier L. C denotes the price of a standard call. P denotes the price of a standard put. BP (GP) is the price of a European bynary (gap) put option, BC (GC) is the price of a European bynary (gap) call option. *The risk-free rate and the asset's payout ratio are equal so that the risk-neutral drift of the underlying asset price is zero.* The local volatility of the returns on the underlying asset can be time-dependent and price-dependent but must satisfy a logprice-symmetric condition: The volatility of the underlying asset price is a known function $\sigma(S_t, t)$ of the underlying price S_t at time t and it satisfies the symmetry condition$\sigma(S_t,t) = \sigma\left(\frac{S_0^2}{S_t}, t\right)$ for all $S_t \geq 0$ and t in $[0,T]$, where S_0 is the current underlying price. The symmetric condition is satisfied under the Black-Scholes assumptions.

$$C^{L,U}_{knockin}(S_0,K,T) =$$

$$+\underbrace{\left(KU^{-1}C\left(S_0,K^{-1}U^2,T\right)+(U-K)\left(2BC\left(S_0,U,T\right)+U^{-1}C\left(S_0,U,T\right)\right)\right)}_{\text{(single barrier }(U)\text{ up-and-in call with barrier below the strike }(K))}$$

$$+\qquad\underbrace{KL^{-1}P\left(S_0,K^{-1}L^2,T\right)}_{\text{(single barrier }(L)\text{ down-and-in call with barrier below the strike }(K))}$$

$$+\qquad\sum_{n=1}^{\infty}\underbrace{\left(\left(\frac{U}{L}\right)^{+n}KL^{-1}P\left(S_0,\left(\frac{L}{K}\right)L\left(\frac{U}{L}\right)^{-2n},T\right)\right)}_{\text{(single barrier }(L(\frac{U}{L})^{-2n})\text{ down-and-in calls with barrier below the strike }(K(\frac{U}{L})^{-2n}))}$$

$$-\qquad\sum_{n=1}^{\infty}\underbrace{\left(\left(\frac{U}{L}\right)^{-n}C\left(S_0,K\left(\frac{U}{L}\right)^{2n},T\right)\right)}_{\text{(single barrier }(L(\frac{U}{L})^{+2n})\text{ up-and-in calls with barrier below the strike }(K(\frac{U}{L})^{+2n})\text{ = standard calls)}}$$

$$+\sum_{n=1}^{\infty}\underbrace{\left(\left(\frac{U}{L}\right)^{-n}\left(\begin{array}{c}KU^{-1}C\left(S_0,\left(\frac{U}{K}\right)U\left(\frac{U}{L}\right)^{+2n},T\right)\\ +(U-K)\left(\begin{array}{c}2e^{+2n(\ln U-\ln L)}\times\\ BC\left(S_0,U\left(\frac{U}{L}\right)^{+2n},T\right)\\ +U^{-1}C\left(S_0,U\left(\frac{U}{L}\right)^{+2n},T\right)\end{array}\right)\end{array}\right)\right)}_{\text{(single barrier }(U(\frac{U}{L})^{+2n})\text{ up-and-in calls with barrier above the strike }(K(\frac{U}{L})^{+2n}))}$$

$$-\sum_{n=1}^{\infty}\underbrace{\left(\left(\frac{U}{L}\right)^{+n}\left(\begin{array}{c}P\left(S_0,K\left(\frac{U}{L}\right)^{-2n},T\right)\\ +(U-K)\left(\begin{array}{c}U^{-1}2GP\left(S_0,U\left(\frac{U}{L}\right)^{-2n},T\right)\\ +U^{-1}C\left(S_0,U\left(\frac{U}{L}\right)^{-2n},T\right)\end{array}\right)\end{array}\right)\right)}_{\text{(single barrier }(U(\frac{U}{L})^{-2n})\text{ down-and-in calls with barrier above the strike }(K(\frac{U}{L})^{-2n}))}$$

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action happens at the lower barrier. Along there, for high logprice volatility levels (50%), the U-in call makes the Basic Portfolio nearly 100% exceed the vanilla call value.

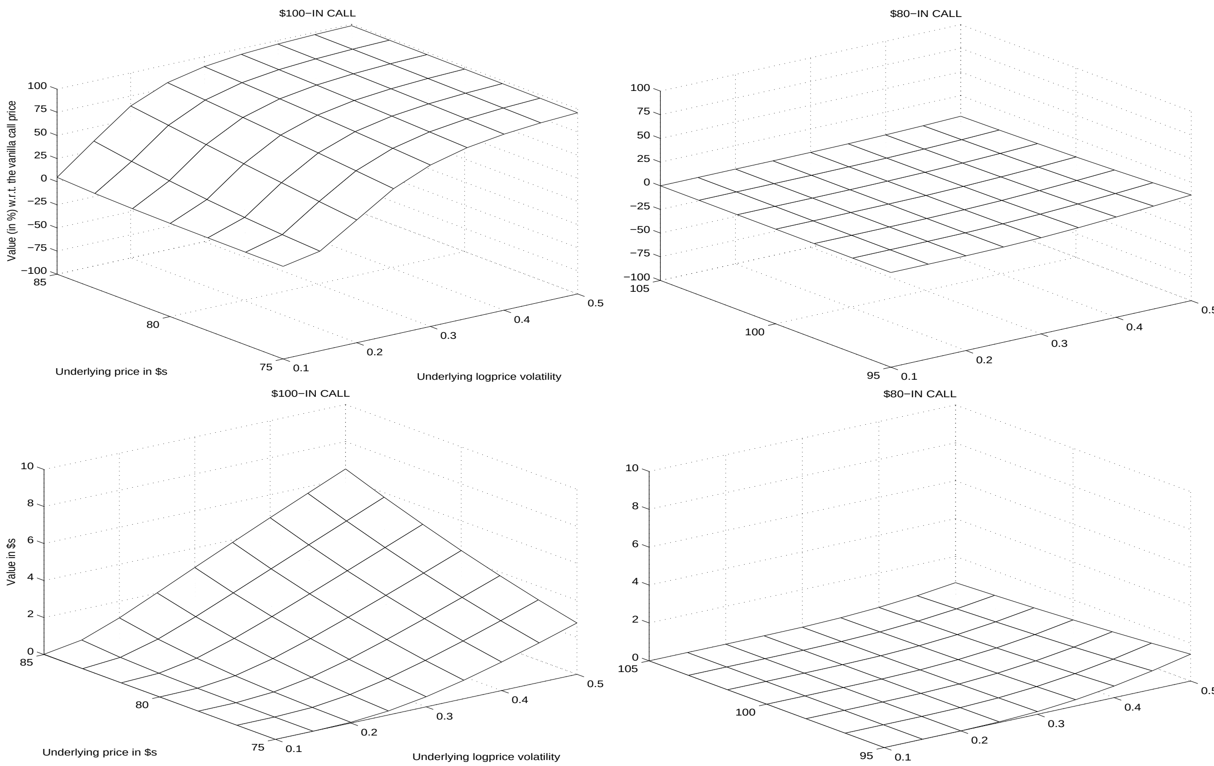

Figure **1**: **Unwanted value contribution along the two barriers.** The \$80-in and \$100-in calls have strike \$90 and 3-month maturity. The Black-Scholes assumptions hold. The cost of carrying the underlying asset (repo rate, 6%, minus asset's payout rate, 3%) is 3%.

The value of UFLI (Parts I and II) eliminates the unwanted value along the upper barrier. Indeed, UFLI is a short position in a L-in call that becomes available as soon as the upper barrier U is hit first before or at maturity, with today's value

$$-E^Q\left(e^{-rT_U}1_{\{T_U<T_L\}}C^L_{knockin}\left(U,K,T-T_U\right)\mid S_0\right).$$

E^Q denotes expectation under the risk-adjusted probability measure and T_U (T_L) is the first time the underlying price reaches the upper barrier U (L). If the lower barrier is hit first, the indicator function calculated on the event $T_U<T_L$ is zero so that there is zero unwanted contribution there.

The value of LFUI (Parts I and II) offsets the unwanted value along the lower barrier. Indeed, LFUI is a short position in a U-in call that becomes available as soon as the lower barrier L is hit first before or at maturity, with today's value

$$-E^Q\left(e^{-rT_L}1_{\{T_L<T_U\}}C^U_{knockin}\left(L,K,T-T_L\right)\mid S_0\right).$$

If the upper barrier is hit first, the indicator function calculated on the event $T_L<T_U$ is zero so that there is zero unwanted contribution there.

CEG, pp. 1174-1176, describe step by step how this architecture works. Consider the replication of the \$80 / \$100 double barrier knockin call. One must zero out unwanted value along each barrier. For example, along \$100, the positive influence of the \$80-in call is offset by selling an amount

$$\frac{m_{BS}\left(0,\ln 100,\ln 80\right)}{m_{BS}\left(0,\ln 80,\ln 100\right)}\times\left(\frac{100}{80}\right)^{-2}=0.6243$$

of up-in calls with barrier $80\left(\frac{100}{80}\right)^{+2}=125$ and strike $90\left(\frac{100}{80}\right)^{+2}=140.63$.

Along \$80, the positive influence of the \$100-in call is offset by selling an amount

$$\frac{m_{BS}\left(0,\ln 80,\ln 100\right)}{m_{BS}\left(0,\ln 100,\ln 80\right)}\times\left(\frac{100}{80}\right)^{+2}=1.6017$$

of down-in calls with barrier $100\left(\frac{100}{80}\right)^{-2}=64$ and strike $90\left(\frac{100}{80}\right)^{-2}=57.60$. However, these short positions generate negative value along the opposite barrier so that other knockin positions must be added. Each additional position hedges at one barrier but creates an error at the other barrier. The size of that error decreases to zero with the number of hedging layers added.

3. The Cost-Of-Carry Effect and The Barrier Effect

How important is keeping track of a drift towards either barrier generated by a non-zero cost of carry of the underlying asset? An answer is the evaluation, along both barriers, of the part of the replicating portfolio proposed by CEG which is meant to be zero over there if the carrying cost had been zero. In Figure **2**, the cost of carry is 3% and the relevant barrier is supposed to have been just hit so that the value of a vanilla call is taken away from both CEG and DBEH values. The CEG and DBEH portfolios consider the first 2 layers of their series-like elements in addition to a non-barrier Basic Portfolio replica and the Basic Portfolio itself respectively. This means that the series terms with $n=1$ and $n=2$ in Tables **I** and **II** are considered. These are the knockin positions involved by the DBEH.

Knockin barrier in \$s	Strike in \$s	Position type	DBEH position nature
244.14	219.73	long	LFUI Part I, $n=2$
195.31	219.73	short	UFLI Part II, $n=2$
156.25	140.63	long	LFUI Part I, $n=1$
125.00	140.63	short	UFLI Part II, $n=1$
(upper barrier) 100.00	90.00	1 unit long	Basic Portfolio
(initial spot price) 90.00	(original strike) 90		
(lower barrier) 80.00	90.00	1 unit long	Basic Portfolio
64.00	57.60	short	LFUI Part II, $n=1$
51.20	57.60	long	UFLI Part I, $n=1$
40.96	36.86	short	LFUI Part II, $n=2$
32.77	36.86	long	UFLI Part I, $n=2$

Figure **2** shows that CEG is substantially off zero (its value is falling 50% short of the vanilla call price) only along \$80 and for quite low logprice volatility levels (10%). However, Figure **1** makes clear that, at such volatility levels, there is no vanilla price action at all. For high volatility levels, CEG is zero along both barriers. *Volatility is likely to be high around the barriers so that the cost of carry should have only second-order effects.* These conclusions are stable across option maturity.

DBEH portfolio amounts vary with the logprice volatility whereas CEG portfolio amounts are pegged to $(\frac{100}{80})^{\pm n}$ with $n = 1, 2$. Consider for example $n = 1$. The maximum percentage absolute difference between $\frac{m_{BS}(0,\ln 80,\ln 100)}{m_{BS}(0,\ln 100,\ln 80)}(\frac{100}{80})^2$ and $\frac{100}{80}$ is 60% in the volatility range $[10\%, 50\%]$. This goes up to 140% for $\frac{m_{BS}(0,\ln 100,\ln 80)}{m_{BS}(0,\ln 80,\ln 100)}(\frac{100}{80})^{-2}$ and $(\frac{100}{80})^{-1}$. In both cases, the maximum difference does not fade out to zero as volatility picks up. However, such volatility effect on DBEH portfolio amounts becomes irrelevant when one looks at the portfolio amounts in \$s. Fixing the portfolio-amount volatility does not affect height and shape of the DBEH graphs in Figures **2**, **3**, and **4**. Even fixing two different portfolio-amount volatility levels in UFLI and LFUI (to accomodate some volatility smirk, for example) is basically neutral. Thus, DBEH graphs take portfolio amounts calculated at varying volatility levels within the range $[10\%, 50\%]$, but they well represent also a volatility-static DBEH strategy where the replicating agent fixes portfolio amounts according to her best guess about the along-the-barriers volatility scenarios.

The hedger can project the risk of barrier instruments, and in particular of double barrier ones, on to simple European options. This means that, as soon as either barrier is hit, 'manual' unwinding of the hedge must take place. This exposes barrier option hedgers to underlying market price manipulation and spurious volatility. This can be the case if the counterpart of the barrier option hedger is a hedge fund. Hedge funds typically use the cheapest means to place big, one-way bets. The temptation to nudge prices can be hard to resist if the result will make a big difference to hedge funds' performance, and hence to the fees their managers earn.

Hence originates Taleb's (1998) risk management hint: *avoid hedging discontinuous exposures (barrier instruments) with continuous ones (non-barrier instruments).* I show that, *along the lower barrier, this hint should be taken with a pinch of salt.* In showing this, I focus on barrier breaching that occurs 3 months before options expiration but the results are quite the same across other possible hitting times. Figure **1** points out that much of the unwanted value is contributed by the \$100-in along \$80. In order to exactly offset this, Part II of LFUI generates the necessary negative value. Unfortunately, such value is steeply downward-sloped value (see Figure **4**). Above \$80, additional knockin positions have up-in barriers substancially higher than \$100 and they command insignificant value. Below \$80, Part II of LFUI is made of short positions and dominates Part I of UFLI as, by construction, its down-in barriers are closer to \$80. The underlying price is in that area so that DBEH residual value is quite unstable at any volatility level.

Intuition suggests that CEG and DBEH should be equivalent not only in value but also in terms of sensitivity to option parameters. Indeed, DBEH specializes to CEG when the cost of carry is zero and the cost-of-carry effect is overwhelmed by the volatility effect. By looking at the surfaces in Figure **3**, one can gauge the key sensitivities of the corresponding CEG and DBEH elements, those with respect to the underlying price (delta and gamma) and the one with respect to logprice volatility (vega). Actual equivalence between CEG and DBEH is evident.

Thus, in Figure **2**, the very small and stable value that CEG has around \$80 for high volatility levels comes from the fact that CEG substitutes the \$80-in call with a non-barrier replica. At \$80, as the \$80-in call becomes the desired vanilla call, one would have a residual replicating portfolio with value worringly sensitive to small underlying price changes. Since a non-barrier replica of the \$80-in call remains in the portfolio if not manually unwound, the difference between the replicating portfolio and a vanilla call includes such remaining value and is flattened out to zero in a comfortably large area around the lower barrier \$80 for high volatility levels. Around the lower barrier, high volatility typically captures the jittery market scenario generated by a bearish sentiment and fear of possible price manipulation.

From this analysis, one may milk out a static replicating strategy that optimally mixes CEG and DBEH. The following table summarizes such a mix.

Knockin barrier in \$s	Strike in \$s	Position type	Position's convenient replica (DBEH or CEG)
244.14	219.73	long	LFUI Part I, $n = 2$
195.31	219.73	short	UFLI Part II, $n = 2$
156.25	140.63	long	LFUI Part I, $n = 1$
125.00	140.63	short	UFLI Part II, $n = 1$
100.00	90.00	1 unit long	Basic Portfolio
80.00	90.00	1 unit long	CEG non-barrier replica
64.00	57.60	short	LFUI Part II, $n = 1$
51.20	57.60	long	CEG non-barrier replica
40.96	36.86	short	LFUI Part II, $n = 2$
32.77	36.86	long	CEG non-barrier replica

This has been written under the assumption that, outside the double barrier corridor, investors prefer the single barrier exotic instrument to its static non-barrier replica when the latter is made of more than 1 piece. CEG replaces LFUI components with 3-piece replicas. Part II of UFLI already has non-barrier nature as its up-in barriers are lower than its call strikes.

4. Concluding Remarks

Barrier derivatives are becoming increasingly liquid. Double barrier options provide investors and risk managers with cheaper means to place bets and to hedge their exposures respectively without paying for the price ranges which they believe unlikely to occur. Double barrier options stipulate a double barrier price corridor which encompasses the initial level of the underlying asset price and the options are triggered or terminated whenever the underlying asset price breaches either barrier for the first time before or at maturity.

The mutual dependence of the two barriers makes these options difficult to price. This work represents their price like a weighted sum of well-known single barrier knockin option prices. The mutual dependence of the two barriers also makes these options difficult to hedge. The pricing representation implies a static hedging strategy (the DBEH).

Double barrier hedges offer full protection only if unwound along the barriers. Along, the DBEH has automatic unwinding. Seller of knockin options may appreciate this if they fear artificial/non-artificial spurious volatility. However, along the lower barrier, tests on hedging performance suggest the substitution of the lower-barrier-in call with a non-barrier replica.

After either barrier is hit, static hedges offer additional benefits with respect to dynamic hedges, especially in high volatility scenarios. This is because static hedge values are very smooth and close to the appropriate level in a wide price range around either barrier.

APPENDIX

The underlying asset has cost of carry equal to $r - d$ (r is the constant riskfree rate and d is the asset's payout rate). Its risk-adjusted logprice, $x_t = \ln S_t$, follows a diffusion process with dynamics:

$$dx_t = (r - d(x_t))\,dt - \frac{1}{2}\sigma^2(x_t)\,dt + \sigma(x_t)\,dW_t,$$

where W_t is a Standard Brownian Motion and r, d, and σ are time-homogeneous and satisfy the conditions that allow for x_t's existence and uniqueness. Set $\ln L = b^-$, $\ln S_0 = x_0$, $\ln U = b^+$ ($b^- \leq x_0 \leq b^+$), and a finite time horizon (option's maturity), T.

The probability density of x_t's transition from x_0 to x during T, $p(x_0, x, T)$ has time-horizon Laplace transform given by:

$$L(\lambda, x_0, x) = \int_0^\infty \exp(-\lambda T)\, p(x_0, x, T)\, dT, \qquad \lambda \geq 0.$$

Taking time-horizon Laplace transforms simplifies the analysis. The Partial Differential Equation (PDE) dynamics of $p(x_0, x, T)$ turns into an Ordinary Differential Equation (ODE) dynamics. A further simplification comes from that the Convolution Property of Laplace transforms applies here. Because of x_t's Strong Markov Property, a probability density involving first exit times until the time horizon can be written as a convolution of similar densities stopped at the time horizon. A transformed convolution is the product of the transformed densities involved in the convolution.

Proposition 3 *The time-horizon Laplace transform $L(\lambda, x_0, x)$ satisfies the ODE*

$$\frac{1}{2}\sigma^2(x_0)\, L_{x_0 x_0} + \left(\mu(x_0) - \frac{1}{2}\sigma^2(x_0)\right) L_{x_0} - \lambda L = 0, \qquad \text{(Laplace ODE)}$$

where L_{x_0} and $L_{x_0 x_0}$ denote $L(\lambda, x_0, x)$'s first and second derivatives with respect to x_0. $L(\lambda, x_0, x)$ is positive and unique.

Proof. The probability density of x_t's transition is an Itô process and it can be conceived as a conditional expectation, that is, as a local martingale. Thus, its local drift must be zero, which means that the expectation, conditional on x_0, of p's infinitesimal changes is null, $E(dp \mid x_0) = 0$. This is p's backward equation and one gets the Laplace ODE by taking time-horizon Laplace transforms in it. L is positive because p is non-negative in all its arguments and it is unique because of Laplace transforms' uniqueness. ■

The moment generating function of x_t's first exit time through some barrier b,

$$m(\lambda, x_0, b),$$

is related to the Laplace transform of the probability density of x_t's transition from x_0 to b as well as that of the probability density of x_t's transition from b to the same level b.

Proposition 4 *The moment generating function of x_t's first exit time through an upper barrier b^+ (lower barrier b^-), $m\left(\lambda, x_0, b^{\pm}\right)$, satisfies the Laplace ODE with these initial conditions:*

$$m(\lambda, b^{\pm}, b^{\pm}) = 1,$$

$$0 < m\left(\lambda, x_0, b^{\pm}\right) \leq 1. \qquad \text{('Probability Bound I')}$$

The solution to the Laplace ODE is given by

$$m(\lambda, x_0, b^{\pm}) = \frac{L(\lambda, x_0, b^{\pm})}{L(\lambda, b^{\pm}, b^{\pm})}. \qquad \text{(Single Barrier M.G.F.)}$$

The Single Barrier M.G.F.s enjoy the following properties. The Single Barrier M.G.F. $m\left(\lambda, x_0, b^-\right)$ is strictly decreasing in x_0 and the Single Barrier M.G.F. $m\left(\lambda, x_0, b^+\right)$ is strictly increasing in x_0. For finite $c \geq 0$,

$$m(\lambda, x_0, b^+ + c) = m(\lambda, x_0, b^+)m(\lambda, b^+, b^+ + c), \qquad \text{('Strong Markov Up')}$$

$$m(\lambda, x_0, b^- - c) = m(\lambda, x_0, b^-)m(\lambda, b^-, b^- - c), \qquad \text{('Strong Markov Down')}$$

For any $\lambda > 0$ and $x_0 \neq b^{\pm}$,

$$m(\lambda, x_0, b^{\pm}) < m(0, x_0, b^{\pm}) \leq 1. \qquad \text{('Probability Bound II')}$$

Proof. Let $\tau_{b^{\pm}}$ be x_t's first exit time through $b^{\pm}$. $\tau_{b^{\pm}}$'s moment generating function satisfies the Laplace ODE as it is the Laplace transform of $\tau_{b^{\pm}}$'s probability density, which in turn satisfies the backward equation (its local drift is zero). If $x_0 = b^{\pm}$, the first exit time is zero for sure, that is, $\exp\left(-\lambda\tau_{b^{\pm}}\right)$ is constant and equal to 1. This gives the first initial condition. The second condition, 'Probability Bound I', comes from the fact that $\exp\left(-\lambda\tau_{b^{\pm}}\right)$ times $\tau_{b^{\pm}}$'s probability density is not greater than $\tau_{b^{\pm}}$'s probability density and that $m(0, x_0, b^{\pm})$ is the probability of ever reaching the barrier $b^{\pm}$. The result for Single Barrier M.G.F. follows from $L(\lambda, x_0, x)$'s structure and properties. The result also comes from Jamshidian (1997) who makes use of the Strong Markov Property and of the Convolution Property of Laplace transforms. The results for 'Strong Markov Up' and 'Strong Markov Down' follow from $L(\lambda, x_0, x)$'s structure and properties. The Strong Markov Property and of the Convolution Property of Laplace transforms prompt an alternative derivation of them. ■

Let $m^+\left(\lambda, x_0, b^-, b^+\right)$ ($m^-\left(\lambda, x_0, b^-, b^+\right)$) be the moment generating function of x_t's first exit time through the upper barrier b^+ (lower barrier b^-) whithout any passage through the lower barrier b^- (upper barrier b^+). The sum of m^+ and m^- gives the moment generating function of x_t's first exit time through either barrier. m^+ and m^- satisfy the Laplace ODE with these initial conditions:

$$\begin{aligned} m^+\left(\lambda, b^+, b^-, b^+\right) &= 1,\ m^+\left(\lambda, b^-, b^-, b^+\right) = 0, \\ m^-\left(\lambda, b^+, b^-, b^+\right) &= 0,\ m^-\left(\lambda, b^-, b^-, b^+\right) = 1. \end{aligned}$$

This is because, if $x_0 = b^+$, the upper barrier is reached for sure and from the very beginning, without touching the lower barrier b^-. This implies $m^+ = 1$ and $m^- = 0$. The reverse holds for $x_0 = b^-$.

Proposition 5 *If x_t is an Arithmetic Brownian Motion (μ and σ are constants), the moment generating functions $m^{\pm}\left(\lambda, x_0, b^-, b^+\right)$ can be decomposed as follows:*

$$m^+\left(\lambda, x_0, b^-, b^+\right) = \qquad \text{('}m^+\text{'s form')}$$

$$\sum_{n=0}^{\infty}\left(\frac{m\left(0, b^+, b^-\right)}{m\left(0, b^-, b^+\right)}\right)^n m\left(\lambda, x_0, b^+ + 2n\left(b^+ - b^-\right)\right) - \sum_{n=0}^{\infty}\left(\frac{m\left(0, b^-, b^+\right)}{m\left(0, b^+, b^-\right)}\right)^{n+1} m\left(\lambda, x_0, b^+ - 2\left(n+1\right)\left(b^+ - b^-\right)\right),$$

and

$$m^-\left(\lambda, x_0, b^-, b^+\right) = \qquad \text{('}m^-\text{'s form')}$$

$$\sum_{n=0}^{\infty}\left(\frac{m\left(0, b^-, b^+\right)}{m\left(0, b^+, b^-\right)}\right)^n m\left(\lambda, x_0, b^- - 2n\left(b^+ - b^-\right)\right) - \sum_{n=0}^{\infty}\left(\frac{m\left(0, b^+, b^-\right)}{m\left(0, b^-, b^+\right)}\right)^{n+1} m\left(\lambda, x_0, b^- + 2\left(n+1\right)\left(b^+ - b^-\right)\right).$$

Proof. I focus on the 'm^+'s form'. Similar arguments justify the 'm^-'s form'. The operator that generates the Laplace ODE is linear so that an absolutely convergent series of Single Barrier M.G.F.s satisfies it. Absolute convergence is sufficient for a safe reversal of order in the infinite-sum and derivative operations. 'Probability Bound I' and 'Probability Bound II' imply that $m\left(\lambda, b^-, b^+\right)$ times $m\left(\lambda, b^+, b^-\right)$ is less than 1. Thus, the absolutely convergent series

$$+m\left(\lambda, x_0, b^+\right)\sum_{n=0}^{\infty}\left(m\left(\lambda, b^+, b^-\right)m\left(\lambda, b^-, b^+\right)\right)^n - m\left(\lambda, x_0, b^-\right)\sum_{n=0}^{\infty}\left(m\left(\lambda, b^-, b^+\right)m\left(\lambda, b^+, b^-\right)\right)^n m\left(\lambda, b^-, b^+\right)$$

satisfies the Laplace ODE and meets m^+'s two initial conditions. The same preliminary decomposition can be obtained from Jamshidian's (1997) analysis by expanding $\left(1-\frac{L\left(\lambda,b^-,b^+\right)}{L\left(\lambda,b^+,b^+\right)}\frac{L\left(\lambda,b^+,b^-\right)}{L\left(\lambda,b^-,b^-\right)}\right)^{-1}$ in power series. The Arithmetic Brownian Motion hypothesis yields

$$m\left(\lambda, b^+, b^-\right) = m\left(\lambda, b^-, b^+\right)\frac{m\left(0, b^+, b^-\right)}{m\left(0, b^-, b^+\right)}.$$

The Arithmetic Brownian Motion hypothesis implies that the travel distance $\left[b^-, b^+\right]$ can be shifted by any shifting factor $\pm c$. Set c equal to either $n\left(b^+ - b^-\right)$ or $\frac{1}{2}n\left(b^+ - b^-\right)$). Then, 'Strong Markov Up' and 'Strong Markov Down' lead to 'm^+'s form''s actual form. ■

Proof of Proposition 1

The probability density which prices the double knockin contracts has the following option-maturity Laplace transform:

$$m^+ L(\lambda, \ln U, \ln S_T) + m^- L(\lambda, \ln L, \ln S_T).$$

Proposition 5 as well as option prices' homogeneity of degree 1 in the initial price, the strike, and the possible barriers, can be used. This gives the DBEH result and completes the proof. ■

Proof of Proposition 2

The Put Call Symmetry (PCS) states that, under the Black-Scholes assumptions with zero risk-adjusted drift of the underlying price, the value of an amount

$$\frac{1}{\sqrt{\text{call strike}}}$$

of calls is equal to the value of an amount

$$\frac{1}{\sqrt{\text{put strike}}}$$

of puts, if the geometric mean of the call strike and the put strike is the current underlying price:

$$\sqrt{\text{call strike}} \times \sqrt{\text{put strike}} = S_0.$$

By means of the PCS, Bowie and Carr (1994) and Carr, Ellis, and Gupta (1998) show that a European single barrier option can be replicated by a portfolio of European standard calls, European standard puts, and European binary options. A European binary call (put) is a cash-or-nothing option which pays \$1 if the underlying price is above (below) the strike price, and zero otherwise. In particular, they prove the following results for down-in call options, and up-in call options respectively:

$$\begin{aligned}
C^L_{knock-in}\left(S_0, K, T\right) &= KL^{-1}P\left(S_0, K^{-1}L^2, T\right),\ L < K, \\
C^L_{knock-in}\left(S_0, K, T\right) &= P\left(S_0, K, T\right) + (H - K)\begin{pmatrix} 2BP\left(S_0, L, T\right) \\ -L^{-1}P\left(S_0, L, T\right)\end{pmatrix},\ L > K, \\
C^U_{knock-in}\left(S_0, K, T\right) &= KU^{-1}C\left(S_0, K^{-1}U^2, T\right) + (U - K)\begin{pmatrix} 2BC\left(S_0, U, T\right) \\ +U^{-1}C\left(S_0, U, T\right)\end{pmatrix},\ U > K,
\end{aligned}$$

where $P(S_0, K, T)$ is the price of a European standard put option with strike K and maturity T, $BP(S_0, K, T)$ is the price of a European bynary put option, and $BC(S_0, K, T)$ is the price of a European bynary call option. PCS also links European gap put options to European binary put options:

$$GP(S_0, K, T) = K \times BP(S_0, K, T) - P(S_0, K, T),$$

where $GP(S_0, K, T)$ is a European gap put option. A European gap call (put) is a cash-or-nothing option which pays a dollar amount equal to the underlying price if the underlying price is above (below) the strike price, and zero otherwise.

Breaking down the DBEH single barrier components by means of the PCS results and setting the cost of carry, $r - d$, to zero achieves the element-by-element equivalence. ■

REFERENCES

ANDERSEN, L., J. ANDREASEN, and D. ELIEZER (2000): "Static Replication of Barrier Options: Some General Results," Working paper, General Re Financial Products.

BENSON, R. and N. DANIEL (1991): "Up, over and out," *RISK*, 4 (6).

BLACK, F., and M. SCHOLES (1973): "The pricing of options and corporate liabilities," *J. Polit. Econ.*, 81, 637-654.

BOWIE, N., and P. CARR (1994): "Static simplicity," *RISK*, 7 (8).

CARR, P., K. ELLIS, and V. GUPTA (1998): "Static hedging of exotic options," *Journal of Finance*, 53 (3), 1165-1190.

COX, J. and M. RUBINSTEIN (1985), *Option markets*, Prentice Hall, Englewood Cliffs, New Jersey.

DERMAN, N., D. ERGENER and I. KANI (1994): "Forever hedged," *RISK*, 7 (8), 139-145.

DOUADY, R. (1999): "Closed form formulas for exotic options and their lifetime distribution," *International Journal of Theoretical and Applied Finance*, 2 (1), pages 17-42.

DUPONT, D. (2001): "Hedging barrier options: Current methods and alternatives," 2001 EURANDOM Report No. 6.

GEMAN, H., and M. YOR (1996): "Pricing and hedging double-barrier options: a probabilistic approach," *Mathematical Finance*, 4, 365-378.

JAMSHIDIAN, F. (1997): "A note on analytical valuation of double barrier options," working paper, Sakura Global Capital.

HEYNEN, R. C. and H.M. KAT (1994): "Crossing barriers," *RISK*, 7 (June), pp. 46-49, correction (1995), *RISK*, 8 (March), p. 18.

HUDSON, M. (1991): "The value of going out," *RISK*, 4 (3), 183-186.

HUI, H.C. (1996): "One-touch double barrier binary option values," *Applied Financial Economics*, 6, 343-346.

HUI, H.C., LO, C.F., and Yuen, P.H. (2000): "Comment on 'Pricing double barrier options using analytical inversion of Laplace transforms'," *Finance and Stochastics*, 4 (1), pages 95-104.

KUNITOMO, N., and M. IKEDA (1992): "Pricing options with curved boundaries," *Mathematical Finance*, 2, 275-292.

LIN, X.S. (1998): "Double-barrier hitting time distributions with applications to exotic options," *Insurance: Mathematics and Economics*, 23 (1), pages 45-58.

MERTON, R. C. (1973): "Theory of rational option pricing," *Bell J. Econ. Manage. Sci.*, 4, 141-183.

PELSSER, A. (2000): "Pricing double barrier options using analytical inversion of Laplace transforms," *Finance and Stochastics*, 4 (1), pages 95-104.

REINER, E., and M. RUBINSTEIN (1991): "Breaking down the barriers," *Risk*, 4 (8), 28-35.

RICH, D. (1994): "The mathematical foundations of barrier option pricing," *Advances in options and futures*, 7, pp. 267-312.

RITCHKEN, P. (1995): "On pricing barrier options," *The Journal of Derivatives*, 3:2, pp. 19-28.

TALEB, N. (1998): *Dynamic Hedging: Managing Vanilla and Exotic Options,* Prentice Hall, Englewood Cliffs, New Jersey.

TRIPPI, S. (1994): "The mathematics of barrier options," *Advances in options and futures.* 7, pp. 150-172.

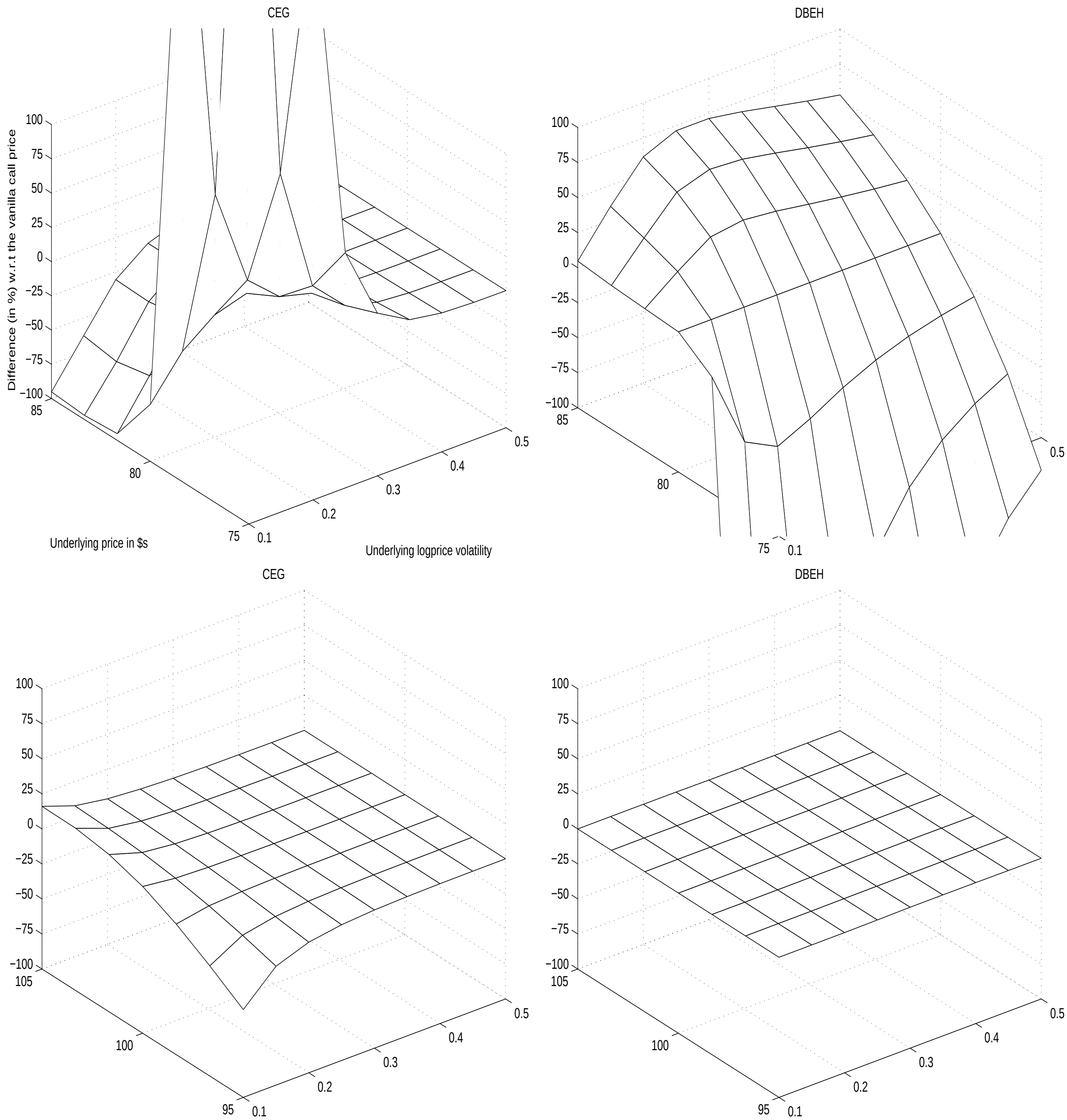

Figure **2**: **The barrier and cost-of-carry effects.** Replicating a \$80 / \$100 double knockin call with strike \$90 and 3-month maturity along the barrier corridor extrema. The Black-Scholes assumptions hold. The cost of carrying the underlying asset (repo rate, 6%, minus asset's payout rate, 3%) is 3%. The CEG and DBEH replicating portfolios consider the first 2 layers of their series-like elements in addition to a Basic Portfolio replica and the Basic Portfolio itself respectively (the Basic Portfolio is a \$80-in plus a \$100-in). *In the graphs, the relevant barrier is supposed to have just been hit (the first among the two barriers) so that the value of a vanilla call is taken away from both CEG and DBEH values.*

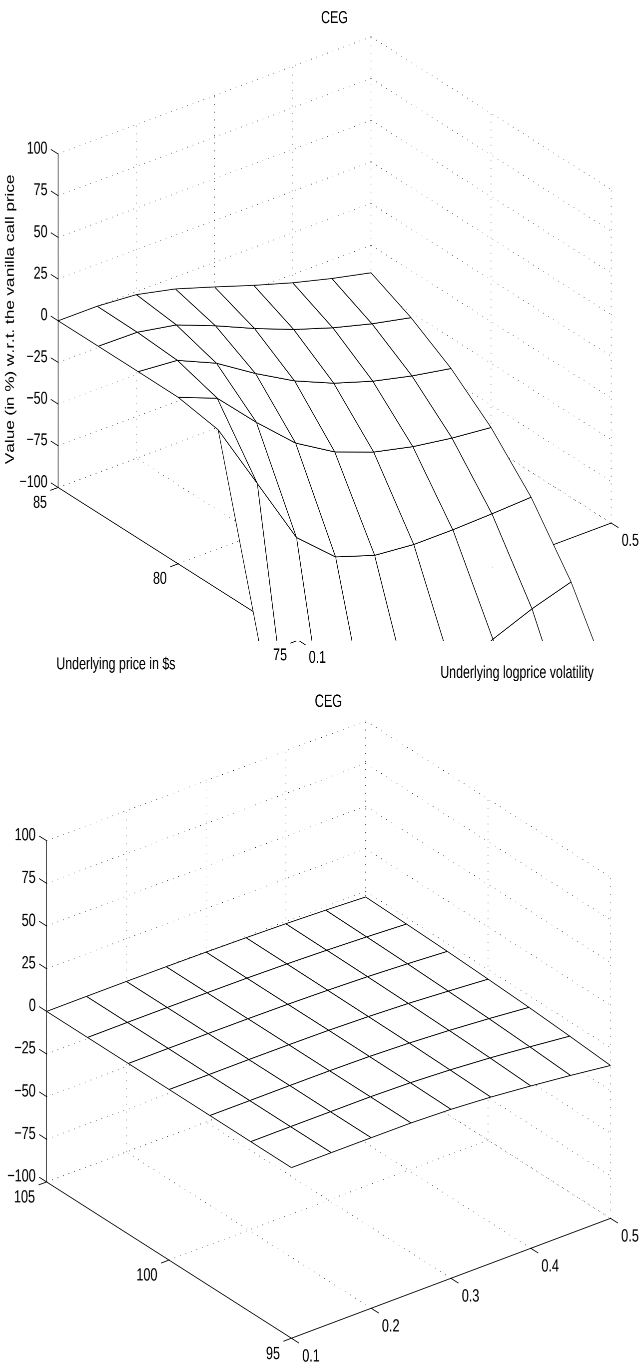

Figure **3**: **CEG and DBEH: Series-like elements.** Around-the-barriers price behaviour of the first 2 layers of the CEG and DBEH series-like elements. They are made of option positions with 3-month maturity. The Black-Scholes assumptions hold. The cost of carrying the underlying asset is 3% (repo rate, 6%, minus asset's payout rate, 3%).

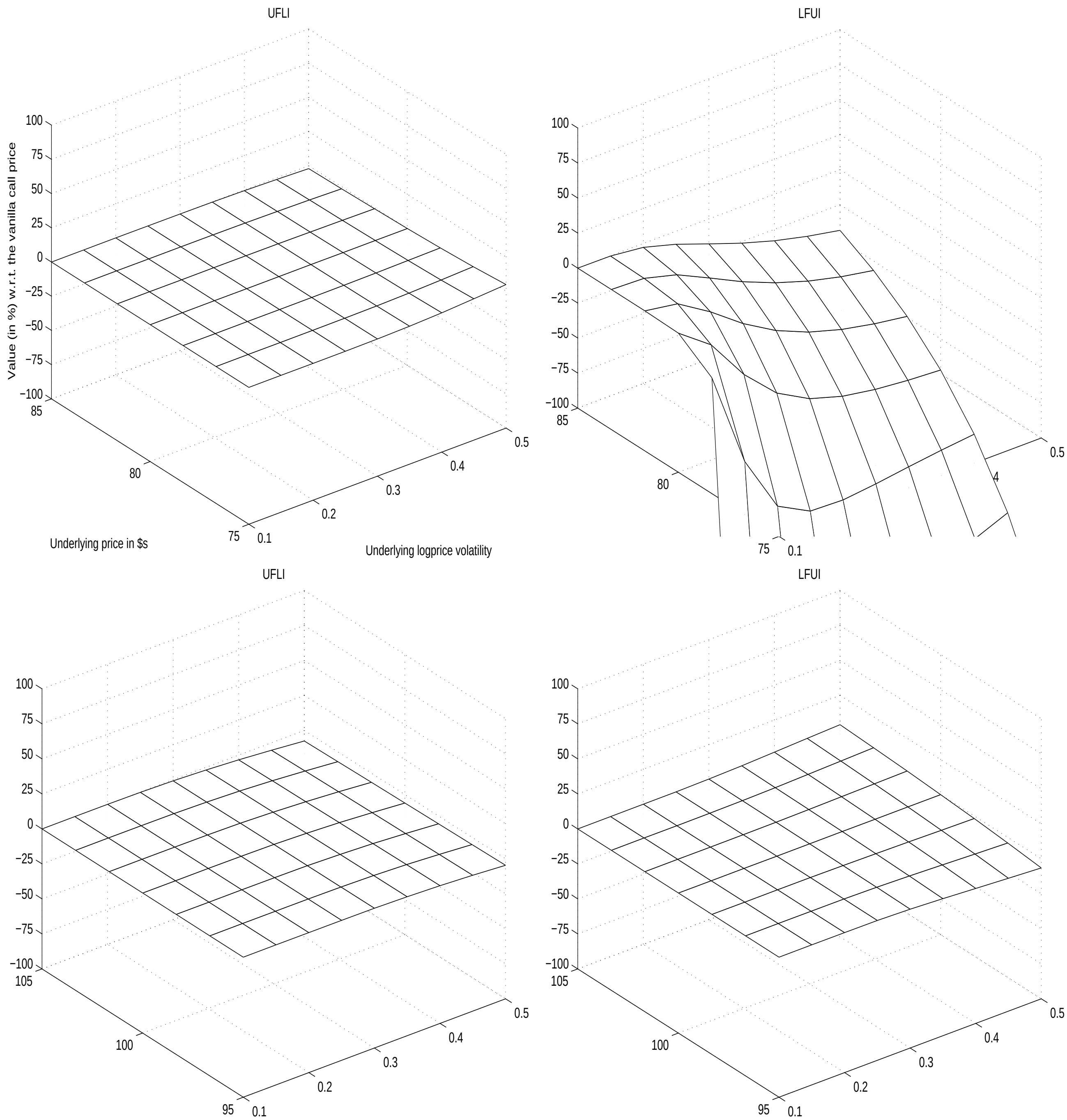

Figure 4: **DBEH: Series-like elements broken into UFLI and LFUI.** Around-the-barriers price behaviour of UFLI and LFUI with 3-month maturity. The Black-Scholes assumptions hold. The cost of carrying the underlying asset (repo rate, 6%, minus asset's payout rate, 3%) is 3%. UFLI and LFUI consider the first 2 layers of their series-like terms.